

NILSYSTEMS AND ERGODIC AVERAGES ALONG PRIMES

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Dedicated to Vitaly Bergelson on the occasion of his 65th birthday

ABSTRACT. A celebrated result by Bourgain and Wierdl states that ergodic averages along primes converge almost everywhere for L^p -functions, $p > 1$, with a polynomial version by Wierdl and Nair. Using an anti-correlation result for the von Mangoldt function due to Green and Tao we observe everywhere convergence of such averages for nilsystems and continuous functions.

1. INTRODUCTION

Ergodic theorems, originally motivated by physics, have found applications in and connections to many areas of mathematics. A prominent example is the result on almost everywhere convergence of ergodic averages along primes by Bourgain [6, 8] (for $p > \frac{1+\sqrt{3}}{2}$) and subsequently Wierdl [38] (for all $p > 1$).

Theorem 1.1. *Let (X, μ, T) be a measure-preserving system, $p > 1$ and $f \in L^p(X, \mu)$. Then the ergodic averages along primes*

$$(1) \quad \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} T^p f$$

converge almost everywhere.

The proof is based on the Carleson transference to the discrete model $(\mathbf{Z}, \text{Shift})$, the Hardy-Littlewood circle method and estimates of prime number exponential sums. An analogous result for polynomials instead of primes was proved by Bourgain [7, 8], see also Thouvenot [37], with variation estimates by Krause [27] showing that the averages converge rapidly. For analogous estimates for averages (1) see Zorin-Kranich [43]. Moreover, Theorem 1.1 has been generalised to polynomials of primes by Wierdl [39] and Nair [32, 33].

Since the proof of Bourgain and Wierdl does not give any information on the set of points where the convergence holds, the following natural question arises.

Question 1.2. *For which systems and functions do the ergodic averages along primes (1) converge everywhere?*

We give a partial answer to this question and show that ergodic averages along polynomials of primes converge everywhere for all nilsystems and all continuous functions. For the definition of a nilsystem and a polynomial sequence see Section 2.

Key words and phrases. Ergodic averages along primes, nilsystems, everywhere convergence.

Theorem 1.3. *Let G/Γ be a nilmanifold, $g : \mathbb{N} \rightarrow G$ be a polynomial sequence and $F \in C(G/\Gamma)$. Then the averages*

$$\frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} F(g(p)x)$$

converge for every $x \in G/\Gamma$. Moreover, if G is connected and simply connected, $g(n) = g^n$ and the system $(G/\Gamma, \mu, g)$ is ergodic, then the limit equals $\int_X F d\mu$.

The key to this result is the powerful theory developed by Green and Tao [21, 22, 20], partially together with Ziegler [23], in their study of arithmetic progressions and linear equations in the primes, in particular the asymptotic orthogonality of the modified von Mangoldt function to nilsequences, see Theorem 2.1 below.

Note that nilsystems and nilsequences has been playing a fundamental role in the study of other kinds of ergodic averages, namely the norm convergence of (linear and polynomial) multiple ergodic averages, motivated by Furstenberg's ergodic theoretic proof [17] of Szemerédi's theorem [36] on the existence of arithmetic progressions in large sets of integers. Here is a list of relevant works: Conze, Lesigne [10], Furstenberg, Weiss [12], Host, Kra [24], Lesigne [31], Ziegler [41], Host, Kra [25], Ziegler [42], Bergelson, Host, Kra [2], Bergelson, Leibman, Lesigne [4], Bergelson, Leibman [3], Leibman [29, 30], Frantzikinakis [13], Host, Kra [26], Chu [9], Eisner, Zorin-Kranich [11], Zorin-Kranich [44]. For other applications of the Green-Tao-Ziegler theory to ergodic theorems see, e.g., Frantzikinakis, Host, Kra [15, 16], Wooley, Ziegler [40], Bergelson, Leibman, Ziegler [5], Frantzikinakis, Host [14].

Our argument is similar to (but simpler than) the one in Wooley, Ziegler [40] in the context of the norm convergence of multiple polynomial ergodic averages along primes.

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2. PRELIMINARIES AND THE W -TRICK

Let G be an s -step Lie group and Γ be a discrete cocompact subgroup of G . The homogeneous space G/Γ together with the Haar measure μ is called an *s -step nilmanifold*. For every $g \in G$, the left multiplication by g is an invertible μ -preserving transformation on G/Γ , and the triple $(G/\Gamma, \mu, g)$ is called a *nilsystem*. Nilsystems enjoy remarkable algebraic and ergodic properties making them an important class of systems in the classical ergodic theory, see Auslander, Green, Hahn [1], Green [19], Parry [34, 35] and Leibman [28]. For example, single and multiple ergodic averages converge everywhere for such systems and continuous functions.

For a continuous function F on G/Γ , the sequence $(F(g^n x))_{n \in \mathbb{N}}$ is called a (*basic linear*) *nilsequence* as introduced by Bergelson, Host, Kra [2]. A nilsequence in their definition is a uniform limit of basic nilsequences (being allowed to come from different systems and functions). Note that the property of Cesàro convergence along primes is preserved by uniform limits, so Theorem 1.3 implies in particular that every nilsequence is Cesàro convergent along primes.

Rather than linear sequences (g^n) , following Leibman [28], Green, Tao [20] and Green, Tao, Ziegler [23], we will consider polynomial sequences $(g(n))$, where $g : \mathbb{N} \rightarrow G$ is called a *polynomial sequence* if it is of the form $g(n) = g_1^{p_1(n)} \cdots g_m^{p_m(n)}$ for some $m \in \mathbb{N}$, $g_1, \dots, g_m \in G$ and some integer polynomials p_1, \dots, p_m . For an abstract equivalent definition see [20]. A sequence of the form $(F(g(n)x))$ for a continuous function F on G/Γ is called a *polynomial nilsequence*. Although this notion seems to be more general than the one of linear basic nilsequences, it is not, see the references at the beginning of the proof of Theorem 1.3 in the following section.

Note that a nilsequence does not determine G , Γ , F etc. uniquely, giving room for reductions. For example, we can assume without loss of generality that $x = \text{id}_G \Gamma$. Moreover, denoting by G^0 the connected component of the identity in G , since we are only interested in the orbit of x under $g(n)$, we can assume without loss of generality that $G = \langle G^0, g_1, \dots, g_m \rangle$.

We use the notations $o_{a,b}(1)$ and $O_{a,b}(1)$ to denote a function which converges to zero or is bounded, respectively, for fixed parameters a, b uniformly in all other parameters.

We now introduce the W -trick as in Green and Tao [21]. Consider

$$\Lambda'(n) := \begin{cases} \log n & \text{if } n \in \mathbb{P}, \\ 0 & \text{otherwise.} \end{cases}$$

For $\omega \in \mathbb{N}$ define

$$W = W_\omega := \prod_{p \in \mathbb{P}, p \leq \omega} p$$

and for $r < W$ coprime to W define the modified Λ' -function by

$$\Lambda'_{r,\omega}(n) := \frac{\phi(W)}{W} \Lambda'(Wn + r), \quad n \in \mathbb{N},$$

where ϕ denotes the Euler totient function.

The key to our result is the following anti-correlation property of $\Lambda'_{r,\omega}$ with nilsequences due to Green and Tao [21] conditional to the “Möbius and nilsequences conjecture” proven by them later in [22]. Here, $\omega : \mathbb{N} \rightarrow \mathbb{N}$ is an arbitrary function with $\lim_{N \rightarrow \infty} \omega(N) = \infty$ satisfying $\omega(N) \leq \frac{1}{2} \log \log N$ for all large $N \in \mathbb{N}$. Note that the corresponding function $W : \mathbb{N} \rightarrow \mathbb{N}$ is then $O(\log^{1/2} N)$.

Theorem 2.1. (Green-Tao [21, Prop. 10.2]) *Let $\omega(\cdot)$ and $W(\cdot)$ be as above, G/Γ be an s -step nilmanifold with a smooth metric, G being connected and simply connected, and let $(F(g^n x))$ be a bounded nilsequence on G/Γ with Lipschitz constant M . Then*

$$\max_{r < W(N), (r, W(N))=1} \left| \frac{1}{N} \sum_{n=1}^N (\Lambda'_{r,\omega(N)}(n) - 1) F(g^n x) \right| = o_{M,G/\Gamma,s}(1)$$

as $N \rightarrow \infty$.

An immediate corollary is the following, cf. Frantzikinakis, Host, Kra [16, p. 5].

Corollary 2.2. *Let G/Γ be an s -step nilmanifold with a smooth metric, G being connected and simply connected, and let $(F(g^n x))$ be a bounded nilsequence on G/Γ with Lipschitz constant M . Then*

$$\max_{r < W, (r, W)=1} \left| \frac{1}{N} \sum_{n=1}^N (\Lambda'_{r, \omega}(n) - 1) F(g^n x) \right| = o_{M, G/\Gamma, s}(1),$$

where one first takes $\limsup_{N \rightarrow \infty}$ and then $\lim_{\omega \rightarrow \infty}$.

Proof. Define for $\omega, N \in \mathbb{N}$

$$a_\omega(N) := \max_{r \in W, (r, W)=1} \left| \frac{1}{N} \sum_{n=1}^N (\Lambda'_{r, \omega}(n) - 1) F(g^n x) \right|$$

and assume that the claimed convergence does not hold. Then there exist $\varepsilon > 0$ and a subsequence (ω_j) of \mathbb{N} so that

$$\limsup_{N \rightarrow \infty} a_{\omega_j}(N) > \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

In particular there exists a subsequence (N_j) of \mathbb{N} such that $a_{\omega_j}(N_j) > \varepsilon$ for every $j \in \mathbb{N}$.

Define now the function $\omega : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\omega(N) := \omega_j \quad \text{if } N \in [N_j, N_{j+1})$$

which grows sufficiently slowly if (N_j) grows sufficiently fast. Then we have

$$a_{\omega(N_j)}(N_j) = a_{\omega_j}(N_j) > \varepsilon$$

contradicting Theorem 2.1 which states $\lim_{N \rightarrow \infty} a_{\omega(N)}(N) = 0$. Note that this argument respects the claimed uniformity in F, g and x . \square

3. PROOF OF THEOREM 1.3

We first need several standard simple facts.

Lemma 3.1. (See, e.g., [15]) *For a bounded sequence $(a_n) \subset \mathbb{C}$ one has*

$$\lim_{N \rightarrow \infty} \left| \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} a_p - \frac{1}{N} \sum_{n=1}^N \Lambda'(n) a_n \right| = 0.$$

Lemma 3.2. *Let $(b_n) \subset \mathbb{C}$ satisfy $b_n = o(n)$ and let $W \in \mathbb{N}$. Then the following assertions hold.*

- (a) *If $\left(\frac{1}{WN} \sum_{n=1}^{WN} b_n\right)_{N \in \mathbb{N}}$ converges, then so does $\left(\frac{1}{N} \sum_{n=1}^N b_n\right)_{N \in \mathbb{N}}$.*
- (b) *If (b_n) is supported on the primes, then*

$$(2) \quad \frac{1}{WN} \sum_{n=1}^{WN} b_n = \frac{1}{W} \sum_{r < W, (r, W)=1} \frac{1}{N} \sum_{n=1}^N b_{Wn+r} + o_W(1).$$

Proof. (a) is clear.

(b) The growth condition implies

$$\frac{1}{WN} \sum_{n=1}^{WN} b_n = \frac{1}{WN} \sum_{r=1}^W \sum_{n=0}^{N-1} b_{Wn+r} = \frac{1}{W} \sum_{r=1}^W \frac{1}{N} \sum_{n=1}^N b_{Wn+r} + o_W(1).$$

If (b_n) is supported on the primes, (2) follows. \square

The following property of connected nilsystems is well known.

Lemma 3.3. *Let $X := G/\Gamma$ be a connected nilsystem with Haar measure μ and $g \in G$. Then (X, μ, g) is ergodic if and only if (X, μ, g) is totally ergodic.*

Proof. Since ergodicity of a nilsystem is equivalent to ergodicity of its Kronecker factor (also called maximal factor-torus, or “horizontal” torus) $G/([G, G]\Gamma)$, see Leibman [28], we can assume without loss of generality that X is a compact connected abelian group.

Let (X, μ, g) be ergodic, $m \in \mathbb{N}$ and let $F \in L^2(X, \mu)$ be an g^m -invariant function, i.e., $F(g^m x) = F(x)$ for every $x \in X$. Consider the Fourier decomposition

$$F = \sum_{\chi \in \widehat{X}} c_\chi \chi.$$

By the assumption we have

$$F = \sum_{\chi \in \widehat{X}} c_\chi (\chi(g))^m \chi.$$

By the uniqueness of the decomposition we obtain

$$c_\chi = c_\chi (\chi(g))^m \quad \forall \chi \in \widehat{X}.$$

Assume that $c_\chi \neq 0$. Then $(\chi(g))^m = 1$, i.e., $\chi(g)$ is an m^{th} root of unity. Since (X, μ, g) is ergodic, $\{g^n : n \in \mathbb{Z}\}$ is dense in X . Since χ is a character and X is connected, $\chi(g)$ has to be equal to 1 - otherwise X would have two clopen components $\{g^n : m_0 | n\}$ and $\{g^n : m_0 \nmid n\}$, where m_0 is the smallest period of $\chi(g)$. Thus $F = c_1 1$ and (X, μ, g) is totally ergodic. \square

Proof of Theorem 1.3. As mentioned above, we can assume that $x = \text{id}_G \Gamma \in G^0$, where G^0 is the connected component of the identity in G , and $G = \langle G^0, g_1, \dots, g_m \rangle$.

Every polynomial nilsequence can be represented as a linear nilsequence on a larger nilmanifold, see Leibman [28, Prop. 3.14], Chu [9, Prop. 2.1 and its proof] and, in the context of connected groups, Green, Tao, Ziegler [23, Prop. C.2]. Thus we can assume that $g(n) = g^n$ for some $g \in G$.

By the argument in Wooley, Ziegler [40, p. 17], the nilsequence $(F(g^n x))$ can be written as a finite sum of (linear) nilsequences coming from a connected, simply connected Lie group. Thus we can assume without loss of generality that G is connected and simply connected.

We first assume that F is Lipschitz and define $b_n := \Lambda'(n)F(g^n x)$. To show convergence of

$$(3) \quad \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} F(g^p x),$$

by Lemmata 3.1 and 3.2(a) it is enough to find $W \in \mathbb{N}$ so that

$$(4) \quad \frac{1}{WN} \sum_{n=1}^{WN} b_n$$

is a Cauchy sequence.

Indeed, for every $\omega \in \mathbb{N}$ we have by Lemma 3.2(b)

$$\begin{aligned}
 \frac{1}{WN} \sum_{n=1}^{WN} b_n &= \frac{1}{W} \sum_{r < W, (r, W)=1} \frac{1}{N} \sum_{n=1}^N b_{Wn+r} + o_W(1) \\
 &= \frac{1}{\phi(W)} \sum_{r < W, (r, W)=1} \frac{1}{N} \sum_{n=1}^N \Lambda'_{r, \omega}(n) F(g^{Wn+r} x) + o_W(1) \\
 &= \frac{1}{\phi(W)} \sum_{r < W, (r, W)=1} \frac{1}{N} \sum_{n=1}^N (\Lambda'_{r, \omega}(n) - 1) F(g^{Wn+r} x) \\
 (5) \quad &+ \frac{1}{\phi(W)} \sum_{r < W, (r, W)=1} \frac{1}{N} \sum_{n=1}^N F(g^{Wn+r} x) + o_W(1) \\
 &=: I(N) + II(N) + o_W(1).
 \end{aligned}$$

Let $\varepsilon > 0$ and take a large ω such that $\limsup_{N \rightarrow \infty} |I(N)| < \varepsilon$ which exists by Corollary 2.2. Since the sequence $(F(g^{Wn+r} x))_{n \in \mathbb{N}}$ is Cesàro convergent for every r , see Leibman [28] and Parry [34, 35], there is $N_0 \in \mathbb{N}$ such that $|II(N_1) - II(N_2)| < \varepsilon$ whenever $N_1, N_2 > N_0$. Thus the averages (4) form a Cauchy sequence implying convergence of (3).

Take now $F \in C(G/\Gamma)$ arbitrary, $x \in G/\Gamma$ and $\varepsilon > 0$. By the uniform continuity of F there exists $G \in C(G/\Gamma)$ Lipschitz with $\|F - G\|_\infty \leq \varepsilon$. We then have

$$\begin{aligned}
& \left| \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} F(g^p x) - \frac{1}{\pi(M)} \sum_{p \in \mathbb{P}, p \leq M} F(g^p x) \right| \\
& \leq \left| \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} F(g^p x) - \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} G(g^p x) \right| \\
& \quad + \left| \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} G(g^p x) - \frac{1}{\pi(M)} \sum_{p \in \mathbb{P}, p \leq M} G(g^p x) \right| \\
& \quad + \left| \frac{1}{\pi(M)} \sum_{p \in \mathbb{P}, p \leq M} G(g^p x) - \frac{1}{\pi(M)} \sum_{p \in \mathbb{P}, p \leq M} F(g^p x) \right| \\
& \leq 2\varepsilon + \left| \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} G(g^p x) - \frac{1}{\pi(M)} \sum_{p \in \mathbb{P}, p \leq M} G(g^p x) \right|
\end{aligned}$$

which is less than 3ε for large enough N, M by the above, finishing the argument.

The last assertion of the theorem follows analogously from the decomposition (5) using Lemma 3.3, the fact that a nilsystem is ergodic if and only if it is uniquely ergodic, see Parry [34, 35], and the uniform convergence of Birkhoff's ergodic averages to the space mean for uniquely ergodic systems. The last step (for non-Lipschitz functions) should be modified by showing that the difference $\frac{1}{\pi(N)} \sum_{p \in \mathbb{P}, p \leq N} F(g^p x) - \frac{1}{N} \sum_{n=1}^N F(g^n x)$ converges to zero. \square

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